Combinatorics 2
Review 1: outcomes, outcome spaces and events

Consider the probability of k heads n tosses of a fair coin.

An outcome: a tuple of length n: HTHHTHH......HHT

Outcome space: the set containing all tuples of length n $\Omega^n$

Event: $A$ the set containing all n-tuples with k heads.

$$P(A) = \frac{|A|}{|\Omega|} = \frac{C(n,k)}{2^n}$$
Review 2

Factorial: \( n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1 \)

Permutations: \( P(n, k) = \frac{n!}{(n-k)!} \)

Combinations: \( C(n, k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} \)

The probability of exactly \( k \) heads when flipping a fair coin \( n \) times:

\[
P(A) = \frac{|A|}{|\Omega|} = \frac{C(n, k)}{2^n}\]
Binomial Expansion

\[(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2\]
\[(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3\]

\[(a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3\]

\[(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i\]

Suppose \(a = b = \frac{1}{2}\) then we get:

\[1 = \left(\frac{1}{2} + \frac{1}{2}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{n-i} = \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} \binom{n}{i}\]

Which can also be written as:

\[\sum_{i=0}^{n} \binom{n}{i} = 2^n\]

Which must be the case because ...
Coming back to coin flipping,
How many coin flips do we need to **guarantee** that there are
at least 60 heads
or
at least 60 tails?

\[
\begin{align*}
H & \quad 59 \\
T & \quad 59 \\
2 \times (60 - 1) + 1
\end{align*}
\]
The Pigeon-Hole Principle

There are 3 colors of marbles. How many marbles do we need in order to guarantee that there is at least one color for which we have at least 6 marbles?

\[3 \times (6 - 1) + 1\]
How many people need to be in a room so that at least two of them share a birthday? (assume 365 days in a year)
The Birthday Paradox

How many people do you need in the room so that at least two of them have the same birthday?

For sure? \[3 \leq 6\]

With probability at least half?

Assume all days have the same probability (1/365)

\[K = \text{the number of people in the room.}\]

We want to calculate \(P(A)\) for the event \(A = \{K \text{ birthdays such that at least two are the same}\}\)
How many people do you need in the room so that at least two of them have the same birthday?

\[ A = \left\{ (i_1, i_2, \ldots, i_K), 1 \leq i_j \leq 365 \left| \exists \ 1 \leq j_1 < j_2 \leq K, i_{j_1} = i_{j_2} \right. \right\} \]

Consider the complement,

No two people have the same birthday

\[ A^c = \left\{ (i_1, i_2, \ldots, i_K), 1 \leq i_j \leq 365 \left| \forall \ 1 \leq j_1 < j_2 \leq K, i_{j_1} \neq i_{j_2} \right. \right\} \]

We are using tuples to represent the birthdays, in other words, order is important!

\[ A^c = \{ x \in \Omega, \ x \not\in A \} \]

\[ A^c = \Omega - A \]

A sequence of K birthdates and no 2 have the same birthday

\[ \rightarrow K \text{ days out of } 365 \]

\[ |A^c| = P(365, K) = \frac{365!}{(365 - K)!} \]
\[\Omega = \{1 - 365\}^k \quad |\Omega| = 365^k\]

\[A = \{\text{Tuples where at least one birthday appears more than once}\}\]
\[A^c = \{\text{Tuples where no two birthdays are the same}\}\]

\[|A^c| = P(365, K) = \frac{365!}{(365 - k)!}\]

\[A = \Omega - A^c \quad \Rightarrow \quad |A| = |\Omega| - |A^c|\]

\[Prob(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|}\]

\[Prob(A) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{P(365,k)}{365^k} = 1 - \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \cdots \times \left(\frac{365-k+1}{365}\right)\]
Exercise 1

How many strings contain 3 letters and two digits?

(digits and letters can repeat and there is no restrictions on their order)

Number of ways to combine 3 letters and 2 digits: $\binom{5}{2} 26^3 10^2$

Set of possible 3 letter tuples = $\{A,\ldots,Z\}^3$
The size of this set is $26*26*26 = 26^3$

Set of 2 digits, size of this set is $10*10 = 100$
Excercise 2

What is the number of strings that start with a digit followed by 4 letters, followed by 2 digits?

Answer: this is a product set:

\[ 10 \times 26 \times 26 \times 26 \times 26 \times 10 \times 10 = 26^4 \times 10^3 \]
How many different ways to place 12 pigs into 7 pens, Each bin can hold any number of pigs ?

\[
\binom{12+(7-1)}{7-1} = \binom{18}{6} \in \binom{18}{6}
\]
How many different ways to place 12 pigs into 7 pens, each bin can hold any number of pigs?
How many different ways to place 12 pigs and 7 lions into 7 bins, where each bin can contain any number of pigs and lions?
You have a shelf with 24 books on it. You can pick any 3 books but no two picked books can be right next to each other. How many choices do you have?
24 books:

BLUE: chosen Book
RED: place Holder
If you choose 3 out of 24 books at random, what is the probability that at least 2 would be next to each other?

Size of sample space (Omega):
- number of way to choose 3 out of 24 books: \( P(24,3) \)
- If we don't care about order of chosen books: \( C(24,3) \)

\[
P(A) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{P(22,3)}{P(24,3)} = 1 - \frac{C(22,3)}{C(24,3)}
\]